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ILL-POSED INITIAL-BOUNDARY VALUE PROBLEM FOR A THIRD-ORDER MIXED TYPE EQUATION

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This paper investigates the existence and conditional stability of solutions for a initial-boundary value problem related to a third-order mixed-type Sobolev equation. These types of problems arrays in various fields, including mathematical physics and fluid dynamics, as they model phenomena such as wave propagation in inhomogeneous media and filtration processes. We prove theorems of conditional correctness, another say theorems of uniqueness and conditional stability. Furthermore, the paper presents an approximate solution using regularization methods, demonstrating how to handle the instability inherent in ill-posed problems. Numerical solutions are obtained, with results shown in the form of tables and graphs. The research thus offers valuable insights into solving third-order mixed-type equations, providing a foundation for further exploration in the numerical approximation of improperly posed boundary conditions problems.

Keywords: stability, uniqueness, Sobolev equation, priori estimate, regularization, generalized solution, spectral problem, conditional correctness.

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1 Introduction

Initial-boundary value problems for mixed-category partial differential equations have drawn increasing interest due to their complexity and importance in various physical and engineering applications. These problems, which are often characterized by non-uniqueness and instability in solutions, arise in processes such as wave propagation, fluid flow, and heat transfer in inhomogeneous media. In recent years, several studies have expanded the understanding of ill-posed problems, especially in connection with non-classical equations. The theory of boundary value problems for mixed-type equations, featuring variable coefficients and a changing type manifold, has been a focus of research S.L. Sobolev [1], M.M. Lavrent'ev, L.Ya. Saveliev [2], M.S. Salakhitdinov, T.D. Djuraev, K.S. Fayazov [8], V.N. Vragov, A.I. Kozhanov, K.B. Sabitov, I.O. Khajiev, Y.K. Khudayberganov and many others.

Boundary value problems for parabolic equations have been examined by various researchers, including E.M. Landis, S.P. Shishatsky, and problems of elliptic type equations were investigated by M.M. Lavrent'ev, L.Ya. Saveliev [2] and others. It is important to cite the contributions of S.G. Krein, H.A. Levine [3], and others, who examined boundary value problems for abstract differential-operator equations. In these works have been proof uniqueness of the solution and get estimates of the conditional stability abstract type problems.

A number of distinguished scholars, including A.L. Bukhgeim, V. Isakov, M. Klibanov, and K.S. Fayazov, have studied ill-posed boundary value problems. In particular, the works of K.S. Fayazov [8], K.S. Fayazov with I. O. Khajiev [10, 11], and K.S. Fayazov with Y.K. Khudayberganov [12, 13], concentrated on constructing approximate solutions for non-standard equations..

In our paper we proof the uniqueness and conditional stability of solution for a third-order non-classic equetion. Investigated by us problem belongs to the field ill-posed problems of mathematical physics. In our case the solution of our problem not depend continuously on the initial conditions. By applying regularization methods, we construct approximate solutions and validate them through numerical experiments. These results provide insights into the stability of ill-posed boundary value problems and offer practical techniques for addressing challenges in fields such as thermal physics and fluid dynamics.

2 Problem statement

We study the equation

$$u_{xxt}(x,y,t) + sign(y)u_{yy}(x,y,t) = 0 \tag{1}$$

in the domain $\Omega = \Omega_0 \times Q$, where $\Omega_0 = \{x, y\,|(-\pi;\pi)\times(-1;1)\,, y \neq 0\}\,,\ Q = (0;T)\,,$ $T < \infty$.

Find a solution of equation (1) in the domain Ω so that the initial

$$u(x,y,t)\,|_{t=0} = \rho(x,y)\ , \quad (x,y) \in [-\pi;\pi]\times[-1;1]\,, \tag{2}$$

boundary

$$\begin{aligned} u(x,y,t)\Big|_{\substack{x=-\pi\\x=+\pi}} &= 0,\ (y,t)\in[-1;1]\times\bar{Q},\\ u(x,y,t)\Big|_{\substack{y=-1\\y=+1}} &= 0,\ (x,t)\in[-\pi;\pi]\times\bar{Q} \end{aligned} \tag{3}$$

and gluing

$$\left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=-0} = \left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=+0},\ (x,t)\in[-\pi;\pi]\times\bar{Q} \tag{4}$$

conditions are satisfied, where $i = \overline{0,1}$ and $\rho\,(x,y)$ are given sufficiently smooth functions and satisfied wherein $\rho\,(x,y)\,|_{\partial\Omega_0} = 0$.

2.1 Spectral problem

Find such values of λ for which the following problem has a nontrivial solution:

$$sign\,(y)\,\vartheta_{yy} + \lambda\vartheta_{xx} = 0, (x,y)\in\Omega_0 \tag{5}$$

$$\begin{aligned} \vartheta(x,y)\Big|_{\substack{x=-\pi\\x=+\pi}} &= 0,\ y\in[-1;1],\\ \vartheta(x,y)\Big|_{\substack{y=-1\\y=+1}} &= 0,\ x\in[-\pi;\pi],\\ \left.\frac{\partial^i u(x,y)}{\partial y^i}\right|_{y=-0} &= \left.\frac{\partial^i u(x,y)}{\partial y^i}\right|_{y=+0},\ x\in[-\pi;\pi]\,, \end{aligned} \tag{6}$$

Using the methods of S. G. Pyatkov [5], we can prove that problem (5), (6) has a nondecreasing sequence $\{\lambda_{k,n}^{+}\}_{k,n=1}^{\infty}$, $\{-\lambda_{k,n}^{-}\}_{k,n=1}^{\infty}$ oof eigenvalues and the corresponding eigenfunctions $\{\vartheta_{k,n}^{\pm}(x,y)\}_{k,n=1}^{\infty}$. The eigenvalues $\lambda_{k,n}^{+} = \frac{\mu_k}{n^2}$, $\lambda_{k,n}^{-} = -\frac{\mu_k}{n^2}$ thus correspond to the eigenfunctions

$$\vartheta_{k,n}^{+}(x,y) = X_n(x) \cdot Y_k^{+}(y), \quad \vartheta_{k,n}^{-}(x,y) = X_n(x) \cdot Y_k^{-}(y), \quad k,n \in N,$$

where

$$X_n(x) = \frac{1}{\sqrt{\pi}} \sin(\pi n + nx), \quad n \in N,$$

$$Y_k^{+}(y) = \begin{cases} \sin\sqrt{\mu_k}(y-1)/\cos\sqrt{\mu_k}, \ 0 \leqslant y \leqslant 1, \\ sh\sqrt{\mu_k}(y+1)/ch\sqrt{\mu_k}, \ -1 \leqslant y \leqslant 0, \end{cases} \quad k \in N,$$

$$Y_k^{-}(y) = \begin{cases} sh\sqrt{\mu_k}(y-1)/ch\sqrt{\mu_k}, \ 0 \leqslant y \leqslant 1, \\ \sin\sqrt{\mu_k}(y+1)/\cos\sqrt{\mu_k}, \ -1 \leqslant y \leqslant 0, \end{cases} \quad k \in N.$$

The eigenvalues n^2 and μ_k correspond to the eigenfunctions $X_n(x)$, and $Y_k^{\pm}(y)$, respectively.

The values μ_k, for $k \in N$, are the positive, and satisfied the transcendental equation $tg\alpha = -th\alpha$. Let $\|u\|^2 = (u,u)$, and

$$(u,v) = \int_{\Omega_0} uv d\Omega_0.$$

It is not difficult to see

$$\left(sign(y)\,\vartheta_{k,n}^{+}(x,y), \vartheta_{l,m}^{-}(x,y)\right) = 0, \quad \forall k,n,l,m,$$

$$\left(sign(y)\,\vartheta_{k,n}^{+}(x,y), \vartheta_{l,m}^{+}(x,y)\right) = \begin{cases} 1, & k = l \wedge n = m \\ 0, & k \neq l \wedge n \neq m, \end{cases}$$

$$\left(sign(y)\,\vartheta_{k,n}^{-}(x,y), \vartheta_{l,m}^{-}(x,y)\right) = \begin{cases} -1, & k = l \wedge n = m \\ 0, & k \neq l \wedge n \neq m, \end{cases}$$

where $k,n,l,m \in N$. We introduce norm [see. [5]]

$$\|u(x,y,t)\|_0^2 = \sum_{k,n=1}^{\infty} \left\{ \left| \left(sign(y)\,u(x,y,t), \vartheta_{k,n}^{+}(x,y),\right) \right|^2 + \right. \\ \left. + \left| \left(sign(y)\,u(x,y,t), \vartheta_{k,n}^{-}(x,y),\right) \right|^2 \right\}. \tag{7}$$

Definition. A generalized solution to problem (1)-(4) is a function $u(x,y,t), u_{xy}(x,y,t) \in C(L_2(\Omega)$, that for any arbitrary function $V(x,y,t) \in W_2^{2,2,1}(\Omega)$, $V_{xx}(x,y,T) = 0$, $V(-\pi,y,t) = 0, V(\pi,y,t) = 0, V(x,-1,t) = 0, V(x,1,t) = 0$, satisfies the following integral identity:

$$\int_{\Omega} u(x,y,t)(sign(y)V_{xxt} - V_{yy})d\Omega = -\int_{\Omega_0} sign(y)V_{xx}(x,y,0)\varphi(x,y)d\Omega_0.$$

Let the solution $u(x,y,t)$ of the problem (1)–(4) exists. Then for the equation (1) using the properties of eigenfunctions of the problem (5)-(6) and the definition of a generalized solution, we have

$$\left(u_{k,n}^{\pm}(t)\right)_t - \lambda_{k,n}^{\pm} u_{k,n}^{\pm}(t) = 0 \tag{8}$$

with the initial condition

$$u_{k,n}^{\pm}(0) = \rho_{k,n}^{\pm}\,,\ \ k,n \in N\,, \tag{9}$$

where

$$\rho_{k,n}^{\pm} = \left(sgn(y)\rho(x,y), \vartheta_{k,n}^{\pm}(x,y)\right), k,l \in N.$$

3 A priori estimate

Theorem 1. If in the domain Ω the function $u(x,y,t)$ satisfies equation (1) and conditions (2)-(4), then to the solution $u(x,y,t)$ for any $t \in Q$ the estimate

$$\|u(x,y,t)\| \leqslant 4\pi \|u_{xy}(x,y,0)\|^{\frac{T-t}{T}} \|u_{xy}(x,y,T)\|^{\frac{t}{T}} \tag{10}$$

is valied.

Proof. Function $\varphi(t)$ defined by integral

$$\varphi(t) = \int\limits_{\Omega_0} u_{xy}^2 d\Omega_0$$

is continuous and has derivatives of the first and second orders in the form

$$\varphi'(t) = 2\int\limits_{\Omega_0} u_{xy}u_{xyt}d\Omega_0,\ \ \varphi''(t) = 2\int\limits_{\Omega_0} u_{xyt}^2 d\Omega_0 + 2\int\limits_{\Omega_0} u_{xy}u_{xytt}d\Omega_0.$$

Transforming the second term of the expression for $\varphi''(t)$ and using equation (1), we obtain

$$\int\limits_{\Omega_0} u_{xy}u_{xytt}d\Omega_0 = \int\limits_{\Omega_0} u_{yy}u_{xxtt}d\Omega_0 = \int\limits_{\Omega_0} sign(y)u_{xxt}sign(y)u_{yyt}d\Omega_0 =$$

$$= \int\limits_{\Omega_0} u_{xxt}u_{yyt}d\Omega_0 = \int\limits_{\Omega_0} u_{xyt}^2 d\Omega_0.$$

Substituting the resulting expression into $\varphi''(t)$ we have

$$\varphi''(t) = 4\int\limits_{\Omega_0} u_{xyt}^2 d\Omega_0.$$

Let $\psi(t) = \ln\varphi(t)$. Then $\psi'(t) = \frac{\varphi'(t)}{\varphi(t)}$ and due to the Cauchy–Bunyakovsky inequality

$$\psi''(t) = \frac{\varphi''(t)\varphi(t) - (\varphi'(t))^2}{(\varphi(t))^2} =$$

$$= \frac{4\int\limits_{\Omega_0} u_{xyt}^2 d\Omega_0 \int\limits_{\Omega_0} u_{xy}^2 d\Omega_0 - \left(2\int\limits_{\Omega_0} u_{xy}u_{xyt}d\Omega_0\right)^2}{\left(\int\limits_{\Omega_0} u_{xy}^2 d\Omega_0\right)^2} \geqslant 0.$$

From the differential inequality $\psi''(t) \geqslant 0$ it follows

$$\psi(t) \leqslant \psi(0)\frac{T-t}{T} + \psi(T)\frac{t}{T}$$

or

$$\varphi(t) \leqslant (\varphi(0))^{\frac{T-t}{T}} (\varphi(T))^{\frac{t}{T}}.$$

Using the form of function $\varphi(t)$, we have

$$\int\limits_{\Omega_0} u_{xy}^2 d\Omega_0 \leqslant \left(\int\limits_{\Omega_0} u_{xy}^2(x,y,0) d\Omega_0 \right)^{\frac{T-t}{T}} \left(\int\limits_{\Omega_0} u_{xy}^2(x,y,T) d\Omega_0 \right)^{\frac{t}{T}}.$$

From this inequality $\int\limits_{\Omega_0} u^2(x,y,t)d\Omega_0 \leqslant 16\pi^2 \int\limits_{\Omega_0} u_{xy}^2(x,y,t)d\Omega_0$, we have

$$\int\limits_{\Omega_0} u^2(x,y,t) d\Omega_0 \leqslant 16\pi^2 \left(\int\limits_{\Omega_0} u_{xy}^2(x,y,0) d\Omega_0 \right)^{\frac{T-t}{T}} \left(\int\limits_{\Omega_0} u_{xy}^2(x,y,T) d\Omega_0 \right)^{\frac{t}{T}},$$

or

$$\|u(x,y,t)\| \leqslant 4\pi \|u_{xy}(x,y,0)\|^{\frac{T-t}{T}} \|u_{xy}(x,y,T)\|^{\frac{t}{T}},$$

thus (10) is proved.

4 Theorem of uniqueness and conditional stability

Let

$$M = \{u : \|u_{xy}(x,y,T)\| \leqslant m\}. \tag{11}$$

Theorem2. Let a solution of the problem (1)–(4) exists and $u(x,y,t) \in M$. Then the solution of the problem (1)–(4) is unique.

Proof. Let there be two solutions $u_1(x,y,t), u_2(x,y,t)$ to the problem (1)–(4) under consideration. Then their difference $u(x,y,t) = u_1(x,y,t) - u_2(x,y,t)$ is a solution to the problem

$$u_{xxt}(x,y,t) + sign(y) u_{yy}(x,y,t) = 0 \tag{12}$$

$$u(x,y,t)\,|_{t=0} = 0\ ,\ (x,y) \in [-\pi;\pi] \times [-1;1]\,, \tag{13}$$

$$\begin{aligned} u(x,y,t)\Big|_{\substack{x=-\pi \\ x=+\pi}} &= 0,\ (y,t) \in [-1;1] \times \bar{Q}, \\ u(x,y,t)\Big|_{\substack{y=-1 \\ y=+1}} &= 0,\ (x,t) \in [-\pi;\pi] \times \bar{Q}. \end{aligned} \tag{14}$$

Let us prove that the solution to problem (12)-(14) $u(x,y,t)$ is identically equal to zero. To solve problem (12)–(14), according to theorem 1, we have

$$\|u(x,y,t)\| \leqslant 4\pi \|u_{xy}(x,y,0)\|^{\frac{T-t}{T}} \|u_{xy}(x,y,T)\|^{\frac{t}{T}}.$$

Due to the initial conditions (11), it follows $\|u(x,y,t)\| \leqslant 0$. Hence $u(x,y,t) = 0$, i.e. $u_1(x,y,t) \equiv u_2(x,y,t)$ for all $(x,y,t) \in \Omega$. This means that the solution to problem (1)–(4) is unique.

Let $u(x,y,t)$ be the solution of the problem (1) - (4) corresponding to the exact data, and $u_\varepsilon(x,y,t)$ be the solution of the problem (1) - (4) corresponding to the approximate data.

Theorem3. Let the solution to problem (1) - (4) exist and $u(x,y,t), u_\varepsilon(x,y,t) \in M$, in addition $\|\rho(x,y) - \rho_\varepsilon(x,y)\|_{W_2^{1,1}} \leqslant \varepsilon$, then for the function $U(x,y,t) = u(x,y,t) - u_\varepsilon(x,y,t)$ with $t \in Q$ the following inequality is true

$$\|U(x,y,t)\| \leqslant 4\pi(\varepsilon)^{1-\frac{t}{T}}(2m)^{\frac{t}{T}}.$$

Proof. Let $U(x,y,t)$ function be a solution to equation (1) satisfying the boundary conditions and gluing conditions (3) - (4) with initial data $U(x,y,0) = \rho(x,y) - \rho_\varepsilon(x,y)$, and $\|\rho(x,y) - \rho_\varepsilon(x,y)\|_{W_2^{1,1}} \leqslant \varepsilon$. From estimate (10) followed

$$\|U(x,y,t)\| \leqslant 4\pi(\varepsilon)^{1-\frac{t}{T}}(2m)^{\frac{t}{T}}.$$

5 Approximate solution

Let in the problem (1)-(4) $\rho\left(x,y\right) = \varphi\left(x\right)\psi\left(y\right)$. One can present solution of problem in the form

$$u(x,y,t) = \sum_{k,n=1}^{\infty} u_{k,n}^{+}(t)\vartheta_{k,n}^{+}(x,y) + \sum_{k,n=1}^{\infty} u_{k,n}^{-}(t)\vartheta_{k,n}^{-}(x,y)$$

or

$$u(x,y,t) = \sum_{k=1}^{\infty}\left(\psi_k^{+}Y_k^{+}\left(y\right)\sum_{n=1}^{\infty}\varphi_n e^{\frac{\mu_k}{n^2}t}X_n\left(x\right)\right) + \sum_{k=1}^{\infty}\left(\psi_k^{-}Y_k^{-}\left(y\right)\sum_{n=1}^{\infty}\varphi_n e^{-\frac{\mu_k}{n^2}t}X_n\left(x\right)\right)$$

where

$$\varphi_n = \int_{-\pi}^{\pi}\varphi\left(x\right)X_n\left(x\right)dx, \quad \psi_k^{+} = \int_{-1}^{1} sign\left(y\right)\psi\left(y\right)Y_k^{+}\left(y\right)dy,$$

$$\psi_k^{-} = -\int_{-1}^{1} sign\left(y\right)\psi\left(y\right)Y_k^{-}\left(y\right)dy, \ \ k,n \in N,$$

$u_{k,n}^{+}(t)$, $u_{k,n}^{-}(t)$ satisfies the equation (8).

Then an approximate solution of the problem with exact data has the form

$$u^N(x,y,t) = \sum_{k=1}^{N}\left(\psi_k^{+}Y_k^{+}\left(y\right)\sum_{n=1}^{\infty}\varphi_n e^{\frac{\mu_k}{n^2}t}X_n\left(x\right)\right) + \sum_{k=1}^{N}\left(\psi_k^{-}Y_k^{-}\left(y\right)\sum_{n=1}^{\infty}\varphi_n e^{-\frac{\mu_k}{n^2}t}X_n\left(x\right)\right),$$

where N is (N integer number) regularization parameter.

The approximate solution with approximate data has the form

$$u_\varepsilon^N(x,y,t) = \sum_{k=1}^{N}\left(\psi_{\varepsilon k}^{+}Y_k^{+}\left(y\right)\sum_{n=1}^{\infty}\varphi_{\varepsilon n} e^{\frac{\mu_k}{n^2}t}X_n\left(x\right)\right) + \sum_{k=1}^{N}\left(\psi_{\varepsilon k}^{-}Y_k^{-}\left(y\right)\sum_{n=1}^{\infty}\varphi_{\varepsilon n} e^{-\frac{\mu_k}{n^2}t}X_n\left(x\right)\right),$$

where

$$\varphi_{\varepsilon n} = \int\limits_{-\pi}^{\pi} \varphi_\varepsilon (x) X_n (x) dx, \;\; \psi_{\varepsilon k}^{+} = \int\limits_{-1}^{1} sign (y) \psi_\varepsilon (y) Y_k^{+} (y) dy,$$

$$\psi_{\varepsilon k}^{-} = - \int\limits_{-1}^{1} sign (y) \psi_\varepsilon (y) Y_k^{-} (y) dy, \;\; k, n \in N.$$

Let $\|\rho(x,y) - \rho_\varepsilon(x,y)\|_{W_2^{1,1}} \leqslant \varepsilon.$ and $u(x,y,t) \in M.$ Then, for the norm of the difference between the exact and approximate solutions, the inequality is as follows:

$$\begin{gathered} 0.5\left\| u (x,y,t) - u_\varepsilon^N (x,y,t) \right\|_0^2 \leqslant \\ \leqslant \left\| u (x,y,t) - u^N (x,y,t) \right\|_0^2 + \left\| u^N (x,y,t) - u_\varepsilon^N (x,y,t) \right\|_0^2. \end{gathered} \tag{15}$$

Let us estimate the second term on the right-hand side of (15), while we made some elementary transformations, and the conditions for estimating the norm of the difference between exact and approximate data are as follows:

$$\begin{gathered} \left\| u^N (x,y,t) - u_\varepsilon^N (x,y,t) \right\|_0^2 = \\ = \sum_{k=1}^{N} \sum_{n=1}^{\infty} e^{2\frac{\mu_k}{n^2} t} (\rho_{k,n} - \rho_{\varepsilon k,n})^2 + \sum_{k=1}^{N} \sum_{n=1}^{\infty} e^{-2\frac{\mu_k}{n^2} t} (\rho_{k,n} - \rho_{\varepsilon k,n})^2 \leqslant \\ \leqslant e^{2\mu_N t} \sum_{k=1}^{N} \sum_{n=1}^{\infty} \left((\rho_{k,n} - \rho_{\varepsilon k,n})^2 + (\rho_{k,n} - \rho_{\varepsilon k,n})^2 \right) \leqslant e^{2\mu_N t} \varepsilon^2 \end{gathered}$$

or

$$\left\| u^N (x,y,t) - u_\varepsilon^N (x,y,t) \right\|_0^2 \leqslant e^{2\mu_N t} \varepsilon^2.$$

Next, we estimate the first term on the right-hand side of inequality (15) provided that, $u (x,y,t)$, $u^N (x,y,t) \in M$

$$\left\| u (x,y,t) - u^N (x,y,t) \right\|_0^2 = \sum_{k=N+1}^{\infty} \left\{ \psi_k^{+} \right\}^2 \left\{ f_k^{+} (t) \right\}^2 + \sum_{k=N+1}^{\infty} \left\{ \psi_k^{-} \right\}^2 \left\{ f_k^{-} (t) \right\}^2,$$

where

$$\left\{ f_k^{+} (t) \right\}^2 = \sum_{n=1}^{\infty} \varphi_n^2 e^{2\frac{\mu_k}{n^2} t}, \; \left\{ f_k^{-} (t) \right\}^2 = \sum_{n=1}^{\infty} \varphi_n^2 e^{-2\frac{\mu_k}{n^2} t}.$$

We estimate the expression

$$\begin{gathered} \sum_{k=N+1}^{\infty} \left\{ \psi_k^{+} \right\}^2 \left\{ f_k^{+} (t) \right\}^2 = \sum_{k=N+1}^{\infty} \left\{ \psi_k^{+} \right\}^2 \sum_{n=1}^{\infty} \varphi_n^2 e^{2\frac{\mu_k}{n^2} t} \leqslant \\ \leqslant \sum_{k=N+1}^{\infty} \left\{ \psi_k^{+} \right\}^2 e^{\mu_k^2 t} \sum_{n=1}^{\infty} \varphi_n^2 e^{\frac{1}{n^4} t} \leqslant C \sum_{k=N+1}^{\infty} \left\{ \psi_k^{+} \right\}^2 e^{\mu_k^2 t}, \end{gathered} \tag{16}$$

according to the condition

$$\sum_{k=1}^{\infty} \left\{ \psi_k^{+} \right\}^2 e^{\mu_k^2 T} \leqslant m^2, \tag{17}$$

where C is a positive constant. By the Lagrange multiplier method will estimate the following extremum problem. and get

$$\sum_{k=N+1}^{\infty} \left\{\psi_k^+\right\}^2 e^{\mu_k^2 t} \leqslant m^2 e^{\mu_{N+1}^2 (t-T)}.$$

Let $\sum_{k=N+1}^{\infty} \left\{\psi_k^-\right\}^2 \left\{f_k^-(t)\right\}^2 = \gamma(N)$, where $\gamma(N) \to 0$ at $N \to \infty$. Thus,

$$\left\|u(x,y,t) - u^N(x,y,t)\right\|_0^2 \leqslant m^2 e^{\mu_{N+1}^2 (t-T)} + \gamma(N).$$

Summing up the estimates, we have

$$0.5 \left\|u(x,y,t) - u_\varepsilon^N(x,y,t)\right\|_0^2 \leqslant e^{2\mu_N t}\varepsilon^2 + m^2 e^{\mu_{N+1}^2 (t-T)} + \gamma(N).$$

Minimizing the assessment on the right side of $\varepsilon > 0$, we obtain an expression for the regularization parameter N. In this context, m is selected arbitrarily and is generally defined according to the specific model.

6 Results

For the numerical solution of problem (1) - (4) we choose the initial function in the form

$$\rho(x,y) = x(\pi^2 - x^2)y(1-y^2),$$

and the approximate data

$$\rho_\varepsilon(x,y) = x(\pi^2 - x^2)y(1-y^2)(1+\varepsilon).$$

We choose N from the conditions $\inf\limits_{\varepsilon>0} \left(e^{2\mu_N t}\varepsilon^2 + m^2 e^{\mu_{N+1}^2 (t-T)} + \gamma(N)\right)$. One can see $\gamma(N) \approx \frac{1}{N^2}$. As an example, consider

$$m = 1000, \quad T = 1, \quad \varepsilon = 10^{-8}, \ t = 0.1, \quad N = 6,$$
$$m = 500, \quad T = 1, \quad \varepsilon = 10^{-8}, \ t = 0.3, \quad N = 3,$$
$$m = 200, \quad T = 1, \quad \varepsilon = 10^{-4}, \ t = 0.5 \quad N = 2.$$

Here m is chosen arbitrarily, and usually it is determined depending on the particular model. For $m = 200, \quad T = 1, \quad \varepsilon = 10^{-4}, \ t = 0.5 \quad N = 2$ the values of the solutions to the problem are given in Tables 1 and 2, as well as in the figures Figure 1 and 2. The following tables and graphs demonstrate that the numerical values of the approximate solution using exact data are very close to those using approximate data.

Table 1 Approximate solution $u^N(x,y,t)$ from exact data

	$y=-0.66$	$y=-0.41$	$y=-0.25$	$y=0.25$	$y=0.41$	$y=0.66$
$x=-2.87$	0.053784	0.211665	0.528862	2.464606	0.193776	-2.8635
$x=-2.09$	0.093156	0.366615	0.916016	4.268823	0.33563	-4.95974
$x=-0.78$	-0.10757	-0.42333	-1.05772	-4.92921	-0.38755	5.72701
$x=0.78$	0.107568	0.423331	1.057724	4.929213	0.387552	-5.72701
$x=2.09$	-0.09316	-0.36662	-0.91602	-4.26882	-0.33563	4.959736
$x=2.87$	-0.05378	-0.21167	-0.52886	-2.46461	-0.19378	2.863505

Table 2 Approximate solution $u_\varepsilon^N(x,y,t)$ from approximate data

	$y=-0.66$	$y=-0.41$	$y=-0.25$	$y=0.25$	$y=0.41$	$y=0.66$
$x=-2.87$	0.053789	0.211687	0.528915	2.464853	0.193796	-2.86379
$x=-2.09$	0.093166	0.366652	0.916108	4.26925	0.335664	-4.96023
$x=-0.78$	-0.10758	-0.42337	-1.05783	-4.92971	-0.38759	5.727583
$x=0.78$	0.107578	0.423373	1.05783	4.929706	0.387591	-5.72758
$x=2.09$	-0.09317	-0.36665	-0.91611	-4.26925	-0.33566	4.960232
$x=2.87$	-0.05379	-0.21169	-0.52891	-2.46485	-0.1938	2.863791

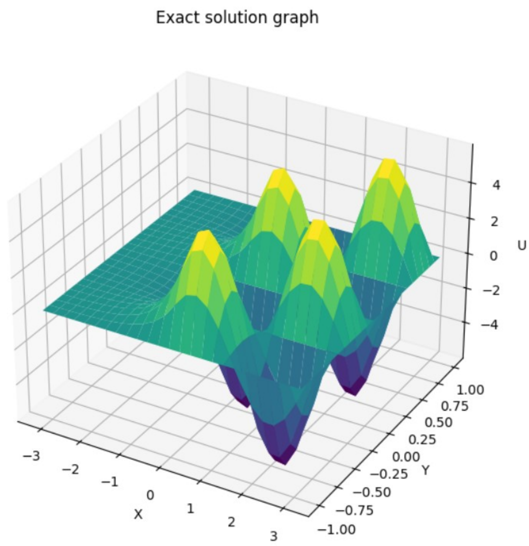

Figure 1 Exact solution graph

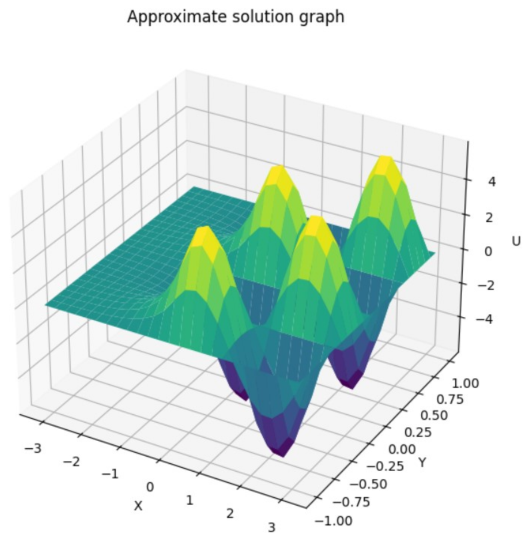

Figure 2 Approximate solution graph

Calculations performed for other values of solutions with other values of parameters, generally speaking, remain within similar limits of accuracy.

7 Conclusions

We using extra information (set of correctness) prove a sovability of non classical problem for Sobolev equation. The conducted studies allow us to construct approximate solutions of these problem on the corresponding correctness sets, which will remain stable with respect to changes in the initial data. As a result, numerical calculations of the solution on a computer are obtained, which are expressed in the form of tables and graphs.

References

[1] Sobolev S.L. 1954. *On a New Problem of Mathematical Physics.* Izv. AN SSSR. Ser. matem., vol. 18, issue 1, – P. 3–50.

[2] Lavrent'ev M.M. and Saveliev L.Ya. 2010. *Theory of Operators and Ill-Posed Problems.* [in Russian], Inst. Mat., Novosibirsk

[3] Levine H.A. 1970. *"Logarithmic convexity, first order differential inequalities and some applications".* Trans. Amer. Math. Soc., 152, – P. 299–320.

[4] Pyatkov S.G. 1988. *"Solvability of boundary value problems for a second-order equation of mixed type [in Russian]," in: Nonclassical partial differential equations.* Collect. Sci. Works, – P. 77–90. Novosibirsk

[5] Pyatkov S.G. 1985. *“On the solvability of a boundary value problem for a parabolic equation with a changing time direction [in Russian],”* Dokl. Akad. Nauk SSSR, 285, No. 6, – P. 1322–1327.

[6] Pyatkov S., Potapkov A., Fayazov K. 2024. *Inverse Problems of Recovering a Source in a Stratified Medium.* J Math Sci 281, – P. 925–937.

[7] Pyatkov S., Soldatov O., Fayazov K. 2023. *Inverse Problems of Recovering the Heat Transfer Coefficient with Integral Data.* J Math Sci 274, – P. 255–268.

[8] Fayazov K.S. 1995. *“An ill-posed boundary-value problem for a second order mixed type equation”.* Uzb. Math. J., 2, – P. 89–93.

[9] Fayazov K., Khudayberganov Y., Pyatkov S. 2023. *Conditional Well-Posedness of the Initial-Boundary Value Problem for a System of Inhomogeneous Mixed Type Equations with Two Degeneration Lines.* J Math Sci 274, – P. 201–214.

[10] Fayazov K.S. and Khajiev I.O. 2015. *“Stability estimates and approximate solutions to a boundary value problem for a fourth order partial differential equation.* [in Russian],” Mat. Zamet. SVFU, 22, – No. 1, – P. 78–88.

[11] Fayazov K.S., Khajiev I.O. 2020. *A nonlocal boundary-value problem for a fourth-order mixed-type equation..* J Math Sci 248, – P. 166–174.

[12] Fayazov K.S. and Khudayberganov Y.K. 2019. *“Ill-posed boundary-value problem for a system of partial differential equations with two degenerate lines.* [in Russian],” Zh. Sib. Fed. Univ., Mat., Fiz., 12, – No. 3, – P. 392–401.

[13] Fayazov K.S. and Khudayberganov Y.K. 2023. *An ill-posed boundary value problem for a mixed type second-order differential equation with two degenerate lines.* Mat. Zamet. SVFU, – P. 51–62.

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НЕКОРРЕКТНАЯ НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ СМЕШАННОГО ТИПА ТРЕТЬЕГО ПОРЯДКА

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В этой статье исследуется единственность и условная устойчивость решения начально-краевой задачи для уравнения Соболева смешанного типа третьего порядка. Данный тип задач встречаются в различных областях, включая математическую физику и гидродинамику, поскольку они моделируют такие явления, как распространение волн в неоднородных средах и процессы фильтрации. Мы доказываем условную корректность задачи, други словами теоремы о единственносты и условной устойчивости. Кроме того, в работе построено приближенное решение задачи методом регуляризации, демонстрирующее, как справиться с неустойчивостью, присущей некорректно поставленным задачам. Получены численные решения, а результаты показаны в виде таблиц и графиков. Таким образом, исследование предлагает ценные идеи для решения уравнений смешанного типа третьего порядка, обеспечивая основу для дальнейшего изучения численного приближения задач с неправильно поставленными граничными условиями.

Ключевые слова: устойчивость, единственность, уравнение Соболева, априорная оценка, регуляризация, обобщенное решение, спектральная задача, условная корректность.

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